M12CH16

TOPIC

16 Vector and Trigonometry

16.1. ORDERED PAIR

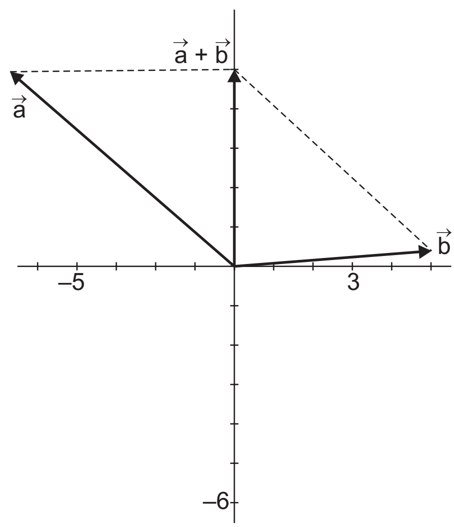
The ordered pair that describes the changes is $(x_2 - x_1, y_2 - y_1)$, in our examples (2 – 0, 5 – 0) or (2, 5). Two vectors are equal if they have the same magnitude and direction. They are parallel if they have the same or opposite direction. We can combine vectors by adding them, the sum of two vectors is called the resultant.

16.2. VECTORS

Vectors are used to represent a quantity that has both a magnitude and a direction. The vector is normally visualized in a graph. A vector between A and B is written as $\overrightarrow{AB}$

The vectors standard position has its starting point in origin.

The component form of a vector is the ordered pair that describes the changes in the x- and y-values. In the graph $x_1 = 0$, $y_1 = 0$, and $x_2 = 2$, $y_2 = 5$. The ordered pair that describes the changes is $(x_2 - x_1, y_2 - y_1)$, in our example (2 – 0, 5 – 0) or (2, 5).



Two vectors are equal if they have the same magnitude and direction. They are parallel if they have the same or opposite direction.

We can combine vectors by adding them, the sum of two vectors is called the resultant.

Example 1. *Add the two following vectors:*

$$\vec{a} = (2, 4),\ \vec{b} = (-1, 6)$$

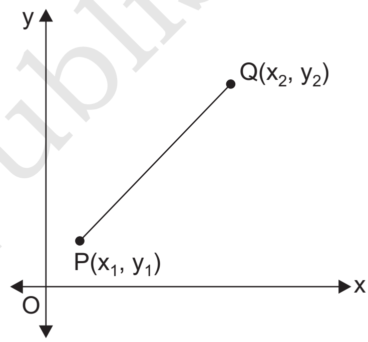
Solution. We add the corresponding components

$$\vec{a} + \vec{b} = (2 + (-1), 4 + 6) = (1, 10)$$

16.3. MAGNITUDE AND DIRECTION OF VECTORS

Magnitude of a Vector

The magnitude of a vector $\overrightarrow{PQ}$ is the distance between the initial point P and the end point Q. In symbols the magnitude of $\overrightarrow{PQ}$ is written as $|\overrightarrow{PQ}|$



If the coordinates of the initial point and the end point of a vector is given, the distance formula can be used to find its magnitude.

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2. *Find the magnitude of the vector $\overrightarrow{PQ}$ whose initial point P is at (1, 1) and end point is at Q is at (5, 3).*

Solution. Use the Distance Formula

$$|\overrightarrow{PQ}| = \sqrt{(5-1)^2 + (3-1)^2} = \sqrt{4^2 + 2^2} = \sqrt{16 + 4}$$

$$= \sqrt{20} \approx 4.5$$

The magnitude of $\overrightarrow{PQ}$ is about 4.5

16.4. WHAT IS THE VECTOR ADDITION

Vectors are represented as a combination of direction and magnitude and are written with an alphabet and an arrow over them (or) with an alphabet written in bold. Two vectors, ***a*** and ***b*,** can be added together using **vector addition**, and the resultant vector can be written as: ***a*** + ***b***.

Before learning about the properties of vector addition, we need to know about the conditions that are to be followed while adding vectors. The conditions are as follows:

- Vectors can be added only if they are of the same nature. For instance, acceleration should be added with only acceleration and not mass.
- We cannot add vectors and scalars together

Consider two vectors **C** and **D.** Where **C** = $C_x i + C_y i + C_z k$ and **D** = $C_x i + D_y j + D_z k$. Then, the resultant vector (or vector sum) **R** **= C + D** = $(C_x + D_x)i + (C_y + D_y)j + (C_z + C_z)k$

Example 3. *Find the addition of vectors* $\overrightarrow{PQ}$ *and* $\overrightarrow{QR}$, *where PQ = (3, 2) and QR = (2, 6).*

Solution. We perform the vector addition by adding their corresponding components

$$PQ + QR = (3, 4) + (2, 6)$$
$$= (3 + 2, 4 + 6) = (5, 10)$$

16.5. WHAT IS THE VECTOR SUBTRACTION

The vector subtraction of two vectors ***a*** and ***b*** is represented by ***a* – *b*** and it is nothing but adding the negative of vector ***b*** to the vector ***a***. *i.e.* ***a* – *b*** = ***a* + (– *b*)**. Thus subtraction of vectors involves the addition of vectors and the negative of a vector. The result of vector subtraction is again a vector. The following are the rules for subtracting vectors

- It should be performed between two vectors only (not between one vector and scalar).
- Both vectors in the subtraction should represent the same physical quantity.

Example 4. *Compute the vector subtraction a – b if a = < 1, – 2, 5 > and b = < 3, – 1, 2>. Also, find its magnitude.*

Solution. Given that $a = < 1, -2, 5 >$ and $b = < 3, -1, 2>$

Now we will find their difference by subtracting the respective components.

$$a - b = < 1, -2, 5 > - <3, -1, 2>$$
$$= < 1 - 3, -2 - (-1), 5 - 2>$$
$$= < -2, -1, 3>$$

Its magnitude is

$$|a - b| = \sqrt{[(-2)^2 + (-1)^2 + 3^2]}$$
$$= \sqrt{(4 + 1 + 9)} = \sqrt{14}$$

16.6. SCALAR MULTIPLICATION OF VECTORS

To multiply a vector by a scalar, multiply each component by the scalar.

If $\vec{u} = (u_1, u_2)$ has magnitude $|\vec{u}|$ and direction d, then $n\vec{u} = n(u_1, u_2) = (n\,u_1, n\,u_2)$ where n is a positive real number, the magnitude is $|n\vec{u}|$, and its direction is d.

Not that if n is negative then the direction of nu is the opposite of d.

Example 5. *Let $u = (-1, 3)$. Find $7u$.*

Solution. $7u = 7(-1, 3)$

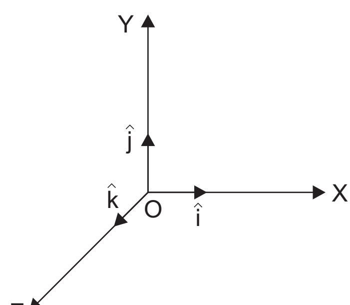
$$= (7(-1), 7(3))$$
$$= (-7, 21)$$

16.7. UNIT VECTOR SYMBOL

Unit Vector is represented by the symbol '^' which is called a cap or hat, such as: $\hat{a}$ it is given by $\hat{a} = \dfrac{a}{|a|}$

Where $|a|$ is for norm or magnitude of vector a.

It can be calculated using a Unit vector formula or by using a calculator.



Unit vector

16.8. UNIT VECTOR FORMULA

As explained above vectors have both magnitude (Value) and direction. They are shown with an arrow.

i.e., $\vec{a}$

And, $\hat{a}$ denotes a unit vector. If we want to change any vector in unit vector, divide it by the vector's magnitude.

Usually, *xyz* coordinates are used to write any vector.

It can be done in two ways:

1. $\vec{a} = (x, y, z)$ using the brackets.
2. $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Formula for magnitude of a vector is:

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector = Vector/Vector's Magnitude

Unit Vector Example

Here is an example based on the unit vector. Observe and follow each step and solve problems based on it.

Example 6. *Find the unit vector* $\vec{p}$ *for the given vector,* $12\hat{i} - 3\hat{j} - 4\hat{k}$ *Show it in both formats – Bracket and Unit vector component.*

Solution. Let's find the magnitude of the given vector first,

$$|p| = \sqrt{x^2 + y^2 + z^2} = \sqrt{144 + 9 + 16} = \sqrt{169} = 13$$

Let's use this magnitude to find the unit vector now:

$$\hat{p} = \frac{p}{|p|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\hat{p} = \frac{12\hat{i} - 3\hat{j} - 4\hat{j}}{13}$$

$$\hat{p} = \frac{12}{13}\hat{i} - \frac{3}{13}\hat{j} - \frac{4}{13}\hat{k}$$

The unit vector in Bracket form is:

$$\hat{p} = \left(\frac{12, -3, -4}{13}\right) = \left(\frac{(12)}{13}, -\frac{(3)}{13}, \frac{(-4)}{13}\right)$$

16.9. POSITION VECTOR

$$(\vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

Where,

$\hat{i}$ = unit vector along *x*-direction

$\hat{j}$ = unit vector along *y*-direction

$\hat{k}$ = unit vector along *z*-direction

16.10. POSITION VECTOR FORMULA

To find the position vector of any a point in the xy-plane, we should first know the point coordinates. Consider two points A and B whose coordinates are (x_1, y_1) and (x_2, y_2) respectively. To determine the position vector, we need to subtract the corresponding components of A from B as follows:

$$AB = (x_2 - x_1)i + (y_2 - y_1)j$$

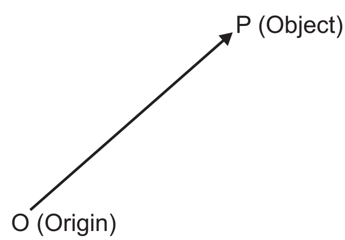
The position vector AB originates from point A and terminates at point B.

16.11. POSITION VECTOR DEFINITION

A position vector is defined as a vector that symbolises either the position or the location of any given point with respect to any arbitrary reference point like the origin. The direction of the position vector always points from the origin of that vector towards a given point.

16.12. POSITIVE VECTOR EXAMPLE

The position vector of an object is measured from the origin, in general. Suppose an object is placed in the space as shown in given figure.



16.13. DIRECTION OF VECTOR

The direction of a vector is the measure of the angle it makes with a horizontal line.

One of the following formulas can be use to find the direction of a vector.

$\tan \theta = \dfrac{y}{x}$, where x is the horizontal change and y is the vertical change.

or $\tan \theta = \dfrac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) is the initial point and (x_2, y_2) is the terminal point.

Example 7. *Find the direction of the vector $\overrightarrow{PQ}$ whose initial point P is at (2, 3) and end point is at Q is at (5, 8).*

Solution. The coordinates of the initial point and the terminal point are given. Substitute them in the formula $\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\tan\theta = \frac{8-3}{5-2} = \frac{5}{3}$$

Find the inverse tan, then use a calculator.

$$\theta = \tan^{-1}\left(\frac{5}{3}\right) \approx 59$$

16.14. WHAT IS STATIC EQUILIBRIUM?

A massive frame hung on a wall using two cables is in static equilibrium. A horizontal beam supported by a strut is in static equilibrium. So what is the definition of static equilibrium, and when do objects fall under this category?

Static equilibrium occurs when an object or a system remains at rest and does not tilt nor rotate. The word "static" means that the body is not in motion, while the term "equilibrium" indicate that all opposing forces are balanced. Thus, a system is in static equilibrium if it is at rest and all forces and other factors influencing the object are balanced.

16.15. STATIC EQUILIBRIUM EXAMPLES

Static equilibrium can be commonly observed in everyday life. Objects at rest are considered systems in static equilibrium, where both net force and net torque are zero. Two examples that demonstrate objects in static equilibrium are:

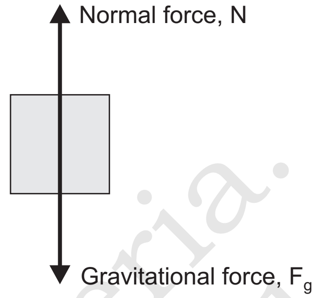
- a book placed on top of a table
- a balanced seesaw

Static Equilibrium

A book placed on top of a table is considered to be in static equilibrium. A free-body diagram, or a diagram showing all the forces acting on the object, can be used to check whether the object satisfies the two conditions of equilibrium.

A book at rest on top of the table is acted upon by the gravitational force and the normal force.

As shown in the diagram, the only forces acting on the book are gravity and normal force. Gravity acts downward, while normal force acts upward, perpendicular to the surface. The forces are also equal in magnitude but opposite in direction. Thus, the net force acting on the book is zero. There is also zero torque acting on the book, satisfying both the first and second conditions of static equilibrium.



16.16. PARALLEL VECTORS

The **parallel vectors** are vectors that have the same direction exactly the opposite direction i.e., for any vector *a*, the vector itself and its opposite vector – *a* are vectors that are always parallel to *a*. Extending this further, any scalar multiple of a is parallel to *a* i.e., a vector a and *ka* are always parallel vectors where '*k*' is scalar (real number).

Example 8. *Find the parallel vector if* $a = 2\hat{i} + 4\hat{j}$ *and* $b = 6\hat{i} + 12\hat{j}$.

Solution. We know that

$$\lambda a = b$$

$$\lambda\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \end{pmatrix}$$

$$2\lambda = 6 \quad \Rightarrow \quad \lambda = 3$$

$$4\lambda = 12 \quad \Rightarrow \quad \lambda = 3$$

λ are same, then vector are parallel.

16.17. WHAT IS AN ORTHOGONAL VECTOR?

In mathematical terms, the word orthogonal means directed at an angle of 90°. Two vectors *u, v* are orthogonal if they are perpendicular, i.e., they form a right angle, or if the dot product they yield is zero.

Example 9. *Find the* orthogonal *vector if A* = (*3, 4, 0*) *and B* = (– *4, 3 ,2*).

Solution. Two vectors A and B are orthogonal, if their dot product is zero. i.e., A · B = 0

$$\begin{aligned} A \cdot B &= A_1 \cdot B_1 + A_2 \cdot B_2 + A_3 \cdot B_3 \\ &= (3) \cdot (-4) + 4 \cdot 3 + 0 \cdot 2 \\ &= -12 + 12 = 0 \end{aligned}$$

So, vector are orthogonal.

16.18. LATITUDES AND LONGITUDES

Latitudes

Equator

- Equator is an imaginary line on the globe that divides it into two equal parts.
- The northern half of the earth is known as the Northern Hemisphere and Southern half is known as the Southern Hemisphere.

Parallels of Latitudes

- Parallels of latitudes are parallel circles from the equator up to the poles.
- They are measured in degrees.

The equator represents the zero degrees latitude. Its distance from the equator to either of the poles is one-fourth of a circle round the earth, it will measure 1/4th of 360 degrees, i.e., 90°. Thus, 90° degrees north latitude marks the North Pole and 90 degrees south latitude marks the South pole.

Latitudes and Longitudes (UPSC Notes)

Important Parallels and Latitudes

- Tropic of Cancer(23½°N) in the Northern Hemisphere.
- Tropic of Capricorn (23½°S) in the Souther Hemisphere.
- Arctic Circle at 66½° north of the equator.
- Antarctic Circle at 66½° south of the equator.

Longitudes

- The meridian which passed through Greenwich, where the British Royal Observatory is located. This meridian is considered as the Prime Meridian.
- Its values is 0° longitude and from it, we count 180° eastward as well as 180° westward. The Prime Meridian and 180° meridian divide the earth into two halves, the Eastern Hemisphere and the Western Hemisphere.

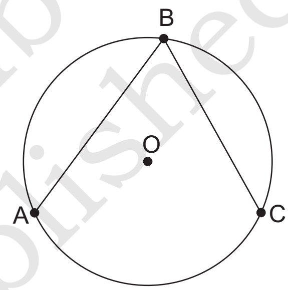
Longitude and Time

- The best means of measuring time is by the movement of the earth, the moon, and the planets. The sun regularly rises and sets every day.

- When the prime meridian of Greenwich has the sum at the highest point in the sky, all the places along this meridian will have mid-day or noon.
- As the earth rotates from west to east, those places east of Greenwich will be ahead of Greenwich time and those to the west will be behind it.

16.19. INSCRIBED ANGLES

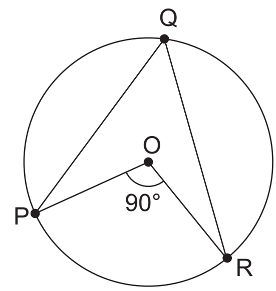
An inscribe angle in a circle is formed by two chords that have a common end point on the circle. This common end point is the vertex of the angle.



Here, the circle with center O has the inscribed angle ∠ABC. The other end points than the vertex, A and C defined the intercepted arc $\overset{\frown}{AC}$ of the circle. The measure of $\overset{\frown}{AC}$ is the measure of its central angle. That is, the measure of ∠AOC.

Example 10. *Find the measure of the inscribed angle ∠PQR.*

Solution. By the inscribed angle theorem, the measure of an inscribed angle is half the measure of the intercepted arc.



The measure of the central angle ∠POR of the intercepter arc $\overset{\frown}{PR}$ is 90°.

Therefore, $m\angle PQR = \frac{1}{2} m\angle POR$

$= \frac{1}{2}(90°) = 45°$

EXERCISE

1. Find the magnitude of the vector $\overrightarrow{PQ}$ whose initial point P is at (2, 2) and end point is at Q is at (6, 8).
2. Find the addition of vectors PQ and QR, where PQ = (4, 3) and QR = (3, 5).
3. Compute the vector subtraction $a - b$ if $a = (1, -2, 5)$ and $b = (3, -1, 2)$.
4. Let $u = (-2, 6)$, find $3u$.
5. Find the unit vector $\vec{q}$ for the given vector, $-2\hat{i} + 4\hat{j} - 4\hat{k}$.
6. Find the unit vector $\vec{q}$ for the given vector, $-3\hat{i} + 9\hat{j} - 4\hat{k}$.

7. Find the direction of the vector $\overrightarrow{PQ}$ whose initial at (3, 4) and end point is at Q is at (6, 9).

8. Find the parallel vector if $a = 4\hat{i} + 8\hat{j}$ and $b = 12\hat{i} + 24\hat{j}$.

9. Find the orthogonal vector if A(4, 5, 0) and B = (– 5, 4, 0).

10. Find $m\angle$LPN.